Journal of Applied Mathematics and Mechanics 73 (2009) 514–523

Contents lists available at ScienceDirect

Journal of Applied Mathematics and Mechanics

journal homepage: www.elsevier.com/locate/jappmathmech

The shapes of bodies with maximum lift-to-drag ratio in supersonic flow☆

V.I. Lapygin*, G.Ye. Yakunina

Moscow, Russia

ARTICLE INFO

Article history:
Received 5 March 2009

ABSTRACT

Assuming that the pressure coefficient on the body surface is defined by the angle between the local normal to it and the velocity vector of the undisturbed flow, the problem of the shape of a body which possesses the maximum lift-to-drag ratio is solved. When the bottom section area and the constant coefficient of friction are given, the optimal body has a plane windward surface positioned at the angle of attack to the undisturbed flow. The leeward surface of the optimal body is parallel to the velocity vector of the undisturbed flow. The absolutely optimal body is a two-dimensional wedge. When additional constraints on the external dimensions of the body are specified, solutions of variational problems are obtained on the basis of which bodies which have the maximum lift-to-drag ratio in supersonic flow are designed.

Local interaction models (LIM) of a supersonic flow with the surface of the body over which the flow occurs have been widely used when calculating the aerodynamic characteristics of supersonic and hypersonic aircraft. Within the limits of LIM, it is assumed that the force due to the action of the flow on an element of the body surface depends solely on the angle between the normal to the element of the surface and the velocity vector of the undisturbed flow. The flow parameters can appear in this relation as constant quantities. A special example of such a model is Newton's drag law.

The tangent wedge method,[1] frequently used in applied investigations, is another example of an LIM which, unlike Newton's formula, takes account of the effect of the Mach number. This method enables one to determine the aerodynamic characteristics of load-carrying bodies with an accuracy comparable with the results of the numerical integration of the equations of motion of an ideal gas.[2] The use of local approaches in solving body optimization problems with respect to one of its integral characteristics (the drag or lift-to-drag ratio) has a long-established history.[3–6] The results of numerous investigations (see Ref. 6) have shown that these solutions, which are found within the limits of LIM, not only provide important qualitative information on the structure of the optimal surface but are often close to the solutions obtained using direct methods, the basis of which is the numerical solutions of the exact equations of supersonic gas flow.

In the case of an arbitrary drag law, written within the limits of LIM, the solution of the problem concerning the spatial shape of the body with minimum drag has been found[7–9] without constraints on the class of admissible surfaces. There is no such solution in the case of the problem of the body with the maximum lift-to-drag ratio and, up to now, it has only been analysed when there are additional constraints imposed on the drag law or the class of admissible surfaces.

For example, a solution of the variational problem of the body with maximum lift-to-drag ratio has been obtained[4,5] within the limits of Newton's model for slender bodies. Using direct methods, solutions of the problem have been found[10,11] for conical bodies without constraints on the body thickness.

The problem of the three-dimensional body with maximum lift-to-drag ratio, without constraints on its thickness and the shape of the optimal surface, is still of current interest. Its analytical solution is found below within the limits of LIM using the basic ideas in Refs 7–9.

1. Formulation of the problem

Considering supersonic flow over a three-dimensional body, we will assume that the unit velocity vector of the free stream $\mathbf{v}$ is parallel to the OX axis of the Cartesian system of coordinates $OXYZ$: $\mathbf{v} = -\mathbf{x}$, where $\mathbf{x}$ is a unit vector of the OX axis (see Fig. 1). The OY axis is directed upwards, and the direction of the OZ axis corresponds to a right-handed system of coordinates. We will assume that the body has a plane

☆ Prikl Mat. Mekh. Vol. 73, No. 5, pp. 717–730, 2009.

* Corresponding author.
E-mail addresses: galina_yakunina@mail.ru, laxenov@yandex.ru (G.Ye. Yakunina).

0021-8928/$ – see front matter © 2009 Elsevier Ltd. All rights reserved.
doi:10.1016/j.jappmathmech.2009.11.004

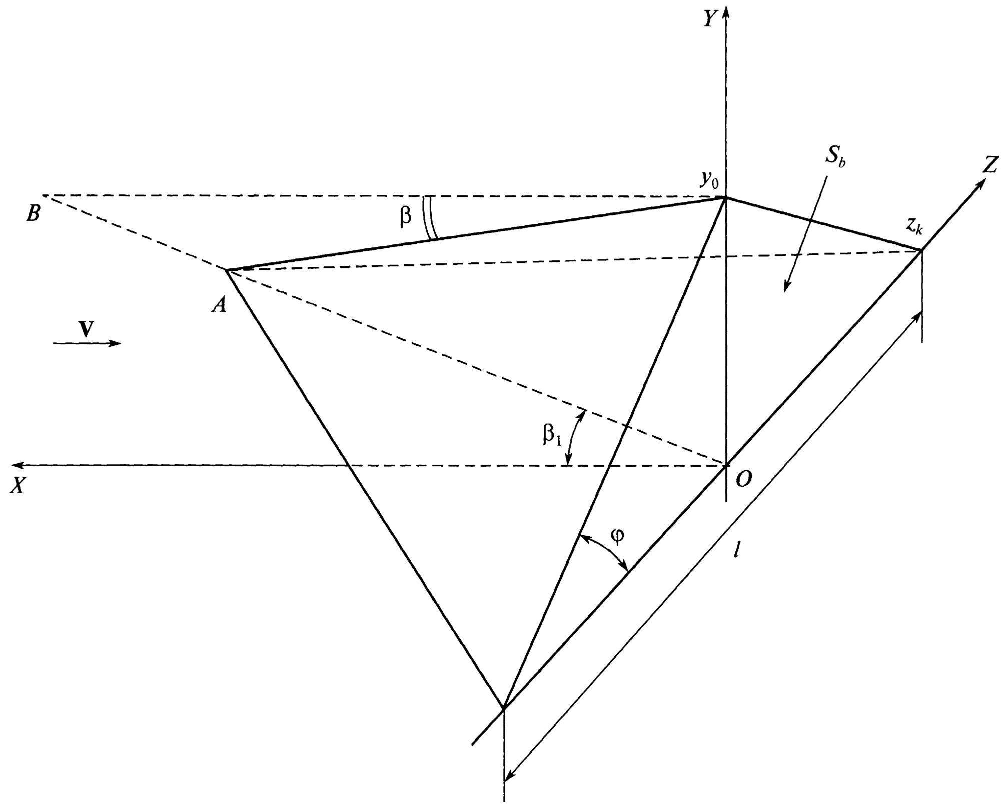

Fig. 1.

base lying in the OYZ plane and that the body surface S facing the flow is a single-valued function of the coordinates of the points of the base: $x = f(y, z)$.

It is assumed that the body interacts with the flow only with the part of the surface on which $\alpha = (\mathbf{n} \cdot \mathbf{v}) \geq 0$, where $\mathbf{n}$ is the unit vector of the inward normal to the element of the body surface, and the pressure coefficient C_p acting on it is a known function of the quantity α and the free-stream Mach number. Within the limits of LIM, in general form we have

$$C_p = C_p(\alpha), \quad \alpha = (\mathbf{n} \cdot \mathbf{v}) \in [0, 1] \tag{1.1}$$

The tangent wedge method[1] for determining the pressure coefficient gives the following relation between C_p and α

$$C_p = \frac{1+\gamma}{2}\left(1 + \left[1 + \left(\frac{4}{(1+\gamma)A}\right)^2\right]^{1/2}\right)\alpha^2; \quad A^2 = (M^2 - 1)\alpha^2 \tag{1.2}$$

Here γ is the ratio of the specific heat capacities and M is the Mach number.

When $A \to \infty$, relation (1.2) corresponds to Newton's formula

$$C_p = (1+\gamma)\alpha^2$$

and, for small values of α and when $A \to 0$, relation (1.2) becomes the formula of the linear theory of supersonic flow over a plane plate

$$C_p = 2(M^2 - 1)^{-1/2}\alpha$$

In the case of the assumptions made, α (1.1) is a function of the coordinates y and z which, on the body surface S facing the flow, is given by the expression

$$\alpha = (1 + u^2 + w^2)^{-1/2}; \quad u = \partial f/\partial y, \quad w = \partial f/\partial z \tag{1.3}$$

We next assume that the coefficient of friction C_f is constant over the body surface and that the shear stresses acting on an element of it lie in the plane of the vectors $\mathbf{n}$ and $\mathbf{v}$ such that the tangential vector τ corresponding to them is coplanar with the vectors $\mathbf{n}$ and $\mathbf{v}$. If $\mathbf{y}$ is the unit vector along the OY axis, then, in this case,

$$(\mathbf{n} \cdot \mathbf{y}) = u\alpha, \quad (\boldsymbol{\tau} \cdot \mathbf{v}) = g, \quad (\boldsymbol{\tau} \cdot \mathbf{y}) = -u\alpha^2/g; \quad g = g(\alpha) = (1 - \alpha^2)^{1/2} \tag{1.4}$$

Taking account of expressions (1.1) and (1.4) as well as the fact that, in the case of the assumptions made, $\alpha > 0$, $dS = \alpha^{-1}dydz$, where dS is the area of an element of the body surface, for the aerodynamic lift coefficient C_y and the drag coefficient C_x one can write the formulae

$$S_bC_y = \iint[C_p(\alpha) - C_f\alpha/g]udydz, \quad S_bC_x = \iint[C_p(\alpha) + C_fg/\alpha]dydz; \quad S_b = \iint dydz \tag{1.5}$$

Henceforth, unless otherwise stated, integration is carried out over the surface of the base of a body having a specified area S_b.

The lift-to-drag ratio of a body is defined by the expression

$$K = C_y/C_x \tag{1.6}$$

The problem of finding the shape of the body possessing the maximum lift-to-drag ratio reduces to determining the function $f(y, z)$ which, for a given area S_b, makes the functional K (1.6) a maximum. In the general case, the functional K depends on the Mach number and the coefficient of friction, which are the parameters of the problem.

We will now write the first variation of the functional K

$$\delta K = (\delta C_y - K\delta C_x)/C_x$$

If K reaches a maximum, then $\delta K = 0$ and the condition

$$\delta(C_y - KC_x) = 0 \tag{1.7}$$

is satisfied.

Taking expression (1.5) into account, it can be seen from this that the problem of a maximum of the functional (1.6) is equivalent to the problem of finding the extremum of the following functional

$$\Phi = \iint F(\alpha, u)dydz$$
$$\Phi(\alpha, u) = [C_p(\alpha) - C_f\alpha/g]u - K[C_p(\alpha) + C_fg/\alpha] + \lambda \tag{1.8}$$

Here λ is a constant factor, and α, g and u are functions of the variables y and z.

2. Analysis of the solution of the variational problem

Taking relations (1.3) and (1.4) into account, it can be seen that the integrand F of the functional (1.8) depends solely on the functions u and w. In this case, the Euler equations for the extremal surface have the form (see Ref. 3)

$$\partial F/\partial u = 0, \quad \partial F/\partial w = 0 \tag{2.1}$$

Using expression (1.8) for F, we rewrite expressions (2.1) in the form of the following system of equations

$$C_p(\alpha) - C_f\alpha/g - Ju\alpha^3 = 0, \quad Jw\alpha^3 = 0$$
$$J = u[C_p'(\alpha) - C_f/g^3] - K[C_p'(\alpha) - C_f/(g\alpha^2)], \quad C_p'(\alpha) = dC_p(\alpha)/d\alpha \tag{2.2}$$

System of equations (2.2), when α>0, has two families of solutions

$$w = 0, \quad C_p(\alpha) - \alpha g^2C_p'(\alpha) - K[C_f - \alpha^2gC_p'(\alpha)]\operatorname{sign}(u) = 0 \tag{2.3}$$
$$J = 0, \quad C_p(\alpha) = C_f\alpha/g \tag{2.4}$$

When $w = 0$, we have $\alpha = (1 + u^2)^{-1/2}$ and, from the second relation of (2.3) we obtain $u = u_1 = \text{const}$. This solution defines a family of planes parallel to the OZ axis

$$x + u_1y + c_1 = 0, \quad c_1 = \text{const} \tag{2.5}$$

which are arranged at an angle of attack β_1 to the flow, where

$$\beta_1 = \arcsin\alpha_1, \quad \alpha_1 = (1 + u_1^2)^{-1/2}$$

Note that $u_1 = \operatorname{ctg}\beta_1$ and, when $u_1 > 0$, segments of the planes (2.5) make a positive contribution to the lift coefficient C_y (1.5) since, when $u_1 < 0$, this contribution is negative.

We find the value $\alpha = \alpha_2 = \text{const}$, from the second equation of (2.4) and, taking this into account and using expression (2.2) for J, from the condition $J = 0$ we obtain $u = u_2 = \text{const}$, which defines the second family of planes

$$x + u_2y + w_2y + c_2 = 0; \quad w_2 = \pm[1 - \alpha_2^2(1 + u_2^2)]^{1/2}/\alpha_2, \quad c_2 = \text{const} \tag{2.6}$$

Each solution (2.6) contains two symmetric planes, touching a cone with an aperture angle $2\arcsin\alpha_2$. It follows from the second condition of (2.4) that the existence, on the surface of the optimal body, of segments of such planes makes no contribution to the lift coefficient C_y (1.5).

System (2.2) has no other solutions when $\alpha > 0$ and, consequently, the optimal body surface must be constructed from segments of planes (2.5) and (2.6). Note that a positive value of K (1.6) in the case of a designed body is only possible in the case when its surface contains segments of the plane (2.5) with $u_1 > 0$.

We will denote the lift-to-drag ratio of a plate on which $w = 0$ by K_1, and put $u = u_1 > 0$. The plate is arranged at an angle of attack β_1 to the flow and is parallel to planes (2.5). Since a segment of plane (2.5) is the sole surface of the optimal body making a positive contribution to the lift coefficient C_y (1.5), then always

$$K < K_1; \quad K_1 = [u_1 C_p(\alpha_1) - C_f]/[C_p(\alpha_1) + C_f u_1] \tag{2.7}$$

In order to obtain the closed surface of the optimal body, it is necessary to combine and join segments of the surfaces (2.5) and (2.6). The necessary Weierstrass-Erdmann condition[3] must be satisfied when they are joined. This condition stipulates that the function F must have the same value on both segments of the surface:

$$F_1 = F_2 \tag{2.8}$$

where F_1 and F_2 are the values of the function F (1.8) for the first family (2.5) and the second family (2.6) of solutions respectively:

$$F_1 = F(\alpha_1, u_1) = u_1 C_p(\alpha_1) - C_f - K(C_p(\alpha_1) + u_1 C_f) + \lambda$$

$$F_2 = F(\alpha_2, u_2) = -KC_f/(\alpha_2 g_2) + \lambda; \quad g_2 = g(\alpha_2)$$

Taking these relations into account, from the equality (2.8) we obtain the expression

$$K = [u_1 C_p(\alpha_1) - C_f]/[C_p(\alpha_1) + u_1 C_f - C_f/(\alpha_2 g_2)]$$

whence it can be seen that condition (2.7) is violated when $C_f \geq 0$. This means that there is no value of K for which conditions (2.7) and (2.8) are simultaneously satisfied. Consequently, the optimal surface cannot simultaneously contain segments of planes (2.5) and (2.6). Since the segments of plane (2.5) with $u_1 > 0$ provide the body with lift, their presence in the structure of the optimal surface is obligatory and segments of surface (2.6) have to be excluded from the treatment. The impossibility of the presence of segments of planes (2.5) with $u_1 < 0$ in the optimal surface can be shown in a similar manner.

As a result, it has been proved that, of the extremals satisfying Euler's equations (2.2), the optimal surface can only contain segments to of plane (2.5) with $u_1 > 0$. It is impossible to construct a closed surface of a body in this case and, consequently, it is impossible to design a body with a maximal lift-to-drag ratio under the assumptions which have been made.

The absence of a solution to the problem is explained in the first place by the fact that formulae (1.5) were used in its formulation to describe the lift-to-drag ratio (1.6), and these formulae were obtained under the assumption that $\alpha > 0$ on the body surface and that there are no segments of the surface with $\alpha = 0$ parallel to the flow.

At the same time, it can be shown that, for a given area S_b, there is always a closed surface with segments $\alpha = 0$, the lift-to-drag ratio of which is greater than that of a body composed of the extremals of (2.5) and (2.6).

A body, constructed from segments of (2.5) and (2.6), which is symmetrical about the OXY plane, has a triangular base and its shape is shown schematically in Fig. 1. It can be seen that, for a specified area S_b, the shape is determined by the angles β_1, β and φ which are found using the parameters u_1, u_2 and w_2:

$$u_1 = \mathrm{ctg}\beta_1, \quad u_2 = -\mathrm{ctg}\beta, \quad w_2 = \pm\mathrm{tg}\varphi/\mathrm{tg}\beta$$

$$y_0 = (S_b \mathrm{tg}\varphi)^{1/2}, \quad z_k = l/2 = (S_b/\mathrm{tg}\varphi)^{1/2}, \quad S_b = y_0 z_k \tag{2.9}$$

Here, y_0 is the height of the body along the OY axis and l is the span of the body along the OZ axis.

Using relations (2.9) and taking account of the fact that conditions (2.4) are satisfied on the segments (2.6), we can write the lift-to-drag ratio in the form

$$K = [u_1 C_p(\alpha_1) - C_f]/[C_p(\alpha_1) + C_f u_1 (1 + \xi)], \quad \xi = [(1 - \alpha_2^2)\cos\varphi]^{-1} \tag{2.10}$$

Without changing the shape of the base and the value of β_1, we will move the vertex of the body from point A to point B. The shape of the body and its parameters are changed: $\beta = 0$, $\alpha_2 = 0$ and the lift-to-drag ratio of the body increase since, in expression (2.10) which remains valid for it, the magnitude of ξ: $\xi = 1/\cos\varphi$ decreases. The body obtained does not belong to the class of admissible shapes since there is a segment with $\alpha = 0$ on its surface but its lift-to-drag ratio is greater than that of a body composed of the extremals of (2.5) and (2.6).

A similar situation arose in the problem of a body of minimum drag for given areas of the base and a wettable surface.[9] The problem was solved then by including segments parallel to the flow, where $\alpha = 0$, in the composition of the optimal surface. A similar approach is used below.

3. Solution of the problem when there are segments of the body surface parallel to the flow

We will assume that there are segments with $\alpha = 0$ on the body surface with an overall area S_0. Their presence makes no contribution to the lift coefficient C_y but it changes the value of the drag such that the lift-to-drag ratio of the body is now given by the expression

$$K = C_y/[C_x + C_f \Delta_0]; \quad \Delta_0 = S_0/S_b \tag{3.1}$$

Here C_y and C_x are the lift and drag coefficients of the segments of the body surface with $\alpha > 0$ for which formulae (1.5) hold.

In this case, the first variation of the functional K has the form $\delta K = [\delta C_y - K\delta C_x - KC_f \delta\Delta_0]/[C_x + C_f \Delta_0]$.
If K reaches a maximum, then $\delta K = 0$ and the condition

$$\delta(C_y - KC_x) - KC_f \delta\Delta_0 = 0 \tag{3.2}$$

is satisfied. Condition (3.2) must be satisfied for any admissible variations in the body surface with maximum lift-to-drag ratio. In particular, it must hold for variations taken at inner points of the surface with $\alpha > 0$ for a constant value of Δ_0: $\delta\Delta_0 = 0$. We obtain from this that, as previously, condition (1.7) must be satisfied on the segments of an optimal body with $\alpha > 0$, and it follows from this condition that the structure of the optimal surface with $\alpha > 0$ is also identical to the structure of the extremals of the functional Φ (1.8) in this case. These extremals must satisfy Euler's equations (2.1) which are given in expanded form as the system of equations (2.2).

Taking account of the results of the analysis of the solutions of system (2.2), we obtain that, when $\alpha>0$, a plane of family (2.5) is the sole surface satisfying the necessary conditions for an extremum and providing lift to the body with maximum lift-to-drag ratio K (3.1). In it, $w = 0$, $u = u_1 = \text{const} > 0$, and it is set at an angle of attack β_1 to the flow which is related to u_1 by the relation $u_1 = \text{ctg}\beta_1$.

On the remaining part of the surface of the optimal body, $\alpha = 0$. It follows from expression (3.10) that this surface must ensure a minimum of the values of Δ_0. Then, $\delta\Delta_0 = 0$, condition (3.2) will be satisfied and the functional K reaches a maximum.

We will assume that the optimal body is symmetrical about the OXY plane. The lower surface of the body is the plane (2.5) with $u = u_1 = \text{const} > 0$ and $c_1 = 0$. In this case, the Oz axis lies in the plane (2.5) and a segment of the axis of length $l = 2z_k$ belongs to the contour of the base of the body (see Fig. 1). In general, the upper surface of the body with $\alpha = 0$ can consist of two parts.

One of the parts is uniquely projected onto the OXZ axis, and its area is given by the expression

$$S_{01} = 2u_1\int_0^{z_k} y(1 + y'^2)^{1/2}dz$$

Here, $y' = dy(z)/dz$, where $y(z)$ is a function of the contour of the base of the body which is a projection of a segment of the upper surface of the body with $\alpha = 0$ onto the OYZ plane.

In the general case, the coordinate $y_k = y(z_k)$ can be equated to zero and then, on the body, it is still part of the surface with $\alpha = 0$, which takes the form of lateral faces from segments of planes parallel to the OXY plane of overall area $S_{02} = u_1 y_k^2$.

As a result, the total area is given by the relation

$$S_0 = S_{01} + S_{02} = u_1\sigma_0; \quad \sigma_0 = 2\left[\int_0^{z_k} y(1 + y'^2)^{1/2}dz + y_k^2/2\right] \tag{3.3}$$

Hence, from the results of the above analysis of the structure of the optimal surface and taking account of relations (1.5), (3.1) and (3.3), we have the expression

$$K = [u_1 C_p(\alpha_1) - C_f]/[C_p(\alpha_1) + u_1 C_f(1 + \xi_0)]; \quad \xi_0 = \sigma_0/S_b \tag{3.4}$$

for the lift-to-drag ratio of the optimal body.

For known flow parameters, the value of K depends on two quantities: u_1 and ξ_0 and, as follows from relations (3.3) and (3.4), their optimal values, which make the functional K (3.4) a maximum, can be sought independently of one another.

Actually, since the area S_b is related to the function $y(z)$ by the expression

$$S_b = 2\int_0^{z_k} y(z)dz \tag{3.5}$$

$\xi_0 \geq 1$ always and an upper limit of the values of K exists:

$$K \leq K_0; \quad K_0 = [u_1 C_p(\alpha_1) - C_f]/[C_p(\alpha_1) + 2u_1 C_f] \tag{3.6}$$

The equality $K = K_0$ holds when $\xi_0 = 1$ when $y' = 0$ and $y(z) \equiv 0$. Consequently, for a given area S_b, the quantity K is the lift-to-drag ratio of a body with an infinite span $l = 2z_k \to \infty$ which in the limit degenerates into a two-dimensional wedge. This wedge consists of two planes, one of which is parallel to the OXZ plane while the second contains the OZ axis and is set at an angle of attack β_1 to the flow: $u_1 = \text{ctg}\beta_1$. The magnitude of K_0 (3.6) is independent of S_b and is determined by the quantity u_1. In the case of a known relation $C_p(\alpha)$, it is possible to find the values of u_1^* and β_1^* for which the magnitude of K_0 is a maximum. The optimal values of the angle of attack of the lower surface of the optimal wedge β_1^* and $K^* = \max K_0$ are presented in Table 1 for different values of C_f and M. Here and henceforth, all calculations are carried out using the relation $C_p(\alpha)$ (1.2) for $\gamma = 1.4$.

We shall say that a wedge with an angle of attack β_1^* and a lift-to-drag ratio K^* is absolutely optimal since it is impossible to design a body with a greater lift-to-drag ratio within the limits of the assumptions made.

However, it is obvious that such a body is of no practical interest and, consequently, for a given area S_b, it is necessary to impose additional constraints on the shape of the unknown body. The solution of the problem with specified constraints on its dimensions is considered below.

Table 1

M	β_1^*, degree			K^*		
	$C_f = 0.001$	0.002	0.003	0.001	0.002	0.003
6	4.32	6.00	7.24	7.345	5.417	4.554
10	5.32	7.19	8.49	6.376	4.834	4.128
15	5.96	7.82	9.11	5.966	4.607	3.970
∞	6.75	8.50	9.72	5.588	4.410	3.836

4. Solution of the problem with additional constraints imposed on the shape of the body

For fixed u_1, a maximum of the quantity K (3.4) is reached when the value of ξ_0 is a minimum. It can be seen from expressions (3.3)–(3.5) that the minimum ξ_0 is independent of u_1 and is reached when the function $y(z)$, defining the upper part of the body, minimizes the functional

$$\Phi_0 = y_k^2/2 + \int_0^{z_k} F_0(y, y')dz, \quad F_0 = y[(1+y'^2)^{1/2} + \lambda_1]$$

Here λ_1 is a constant Lagrange multiplier.

From an analysis of the necessary conditions for a minimum of the functional Φ_0, we obtain that the extremal of $y(z)$ must satisfy Euler's equation[3]

$$F_0 - y'\partial F_0/\partial y' = y[(1+y'^2)^{-1/2} + \lambda_1] = c_1 = \text{const} \tag{4.1}$$

and, in accordance with the Weierstrass–Erdmann conditions,

$$\Delta(F_0 - y'\partial F_0/\partial y') = 0, \quad \Delta(\partial F_0/\partial y') = 0, \quad \partial F_0/\partial y' = yy'(1+y'^2)^{-1/2}$$

it cannot have corner points. Here $\Delta(\ldots)$ denotes the difference of a quantity (...) before and after a corner point.

At the ends of an extremal, the transversality conditions

$$c_1\delta z_k = 0, \quad y_k(y'_k(1+y_0'^2)^{-1/2} + 1)\delta y_k = 0, \quad y_0 y'_0(1+y_0'^2)^{-1/2}\delta y_0 = 0 \tag{4.2}$$

must be satisfied, where $y_0 = y(0)$ and the sign δ denotes the variation of a chosen coordinate.

To obtain a minimum at the points of an extremal, the Legendre condition $y(1+y'^2)^{-3/2} \geq 0$ must be satisfied, and it follows from this that any curve $y(z)$, constructed in accordance with conditions (3.5), (4.1) and (4.2), will minimize the functional Φ_0.

If there are no constraints on the transverse dimensions of the body z_k, it then follows from the first condition of (4.2) that $c_1 = 0$. In this case, we obtain from Eq. (4.1) that $y'^2 = \text{const}$ and, consequently, the upper part of the body is formed by two symmetric planes which are parallel to the flow. However, if y_0 is not given, then, from a combined analysis of conditions (4.1) and (4.2), we obtain the solution $y(z) \equiv 0$ and the optimal body is the two-dimensional wedge considered above, which has a lift-to-drag ratio K_0 (3.6). Such a body is of no practical interest, and, therefore, if there are no constraints on the choice of z_k, y_0 has to be given. Then, $y_k = 0$, and the optimal contour of the base is defined by the linear function

$$y(z) = -tz + y_0, \quad t = y_0^2/S_b \tag{4.3}$$

Hence, when the area of the base S_b and the coordinate y_0 are given, a triangular wing with a wedge-shaped profile and a linear contour of the cross section (4.3), the inclination of which to the OZ axis is defined by the quantity t, will be the optimal body. The shape of the wing can be obtained schematically if, in the case of the body shown in Fig. 1, the vertex is shifted from point A to point B. It follows from formulae (2.9), (3.3)–(3.5) and (4.3) that, in this case,

$$t = y_0/z_k = \text{tg}\varphi, \ \xi_0 = (1+t^2)^{1/2} = 1/\cos\varphi$$

Now suppose constraints are imposed on the transverse dimensions of the body and the value of z_k is given. Then, $\delta z_k = 0$ and, from a joint analysis of relations (4.1) and (4.2), we obtain that the conditions

$$y'_0 = 0, \ y_0 = c_1/(1+\lambda_1), \quad y'_k = -\infty, \ y_k = c_1/\lambda_1 \tag{4.4}$$

are satisfied at the ends of the extremal. It follows from this that $c_1 < 0$, $\lambda_1 < -1$, $y_k > 0$, and Eq. (4.1) takes the form

$$y' = -[y^2/(c_1 - \lambda_1 y)^2 - 1]^{1/2} \tag{4.5}$$

We will assume that all the linear dimensions of the body are divided by z_k. The solution of Eq. (4.5) can then be written in the form

$$z = [\lambda_1(ay^2 + by + c)^{1/2} + c_1 I]/a$$

$$a = 1 - \lambda_1^2, \quad b = 2\lambda_1 a/(1+I^*), \quad c = -a^2/(1+I^*), \quad c_1 = -c/a$$

$$I = |a|^{-1/2}[\pi/2 + \arcsin(|\lambda_1| - (1+I^*)y)], \ I^* = |a|^{-1/2}[\pi/2 + \arcsin(|\lambda_1|^{-1})] \tag{4.6}$$

For known λ_1, all the parameters in expressions (4.4)–(4.6) are uniquely defined. The quantity λ_1 is found from condition (3.5) using the given values of S_b and z_k, which can be rewritten in the form

$$\Delta_k = \frac{S_b}{2z_k^2} = \int_{y_0}^{y_k} z dy + y_k$$

From this, λ_1 is found from the known value of Δ_k using relations (4.4) and (4.6), the dependence of y on z is determined and, together with it, the optimal shape of the contour of the base of the upper part of the body. Optimal contours, constructed for different values of Δ_k are given in Fig. 2. Their shape is independent of u_1, M and C_f and this means that it is independent of the specific form of the function $C_p(\alpha)$ (1.1).

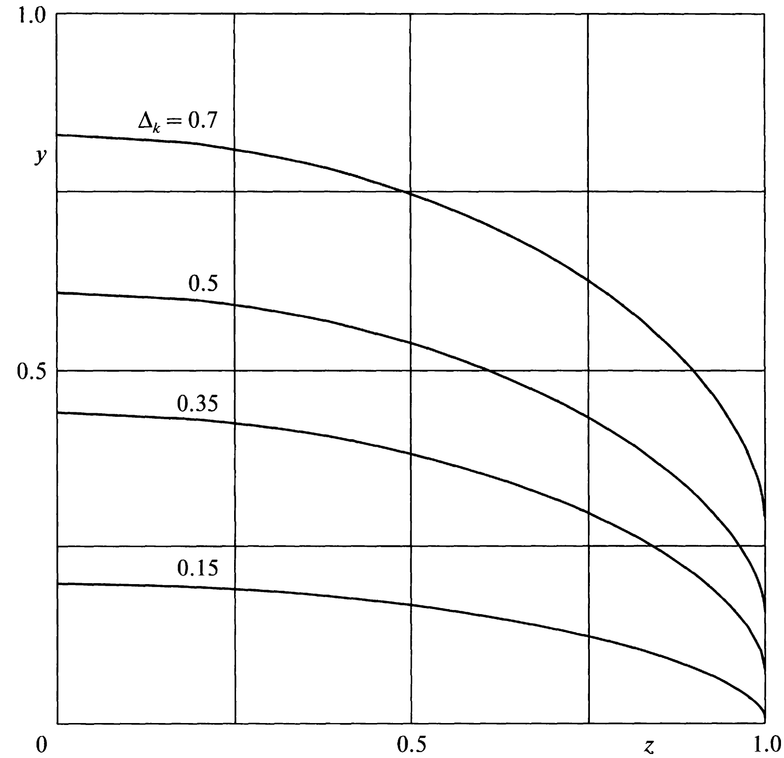
1.0
$\Delta_k = 0.7$
y
0.5
0.5
0.35
0.15
0
0.5
z
1.0

Fig. 2.

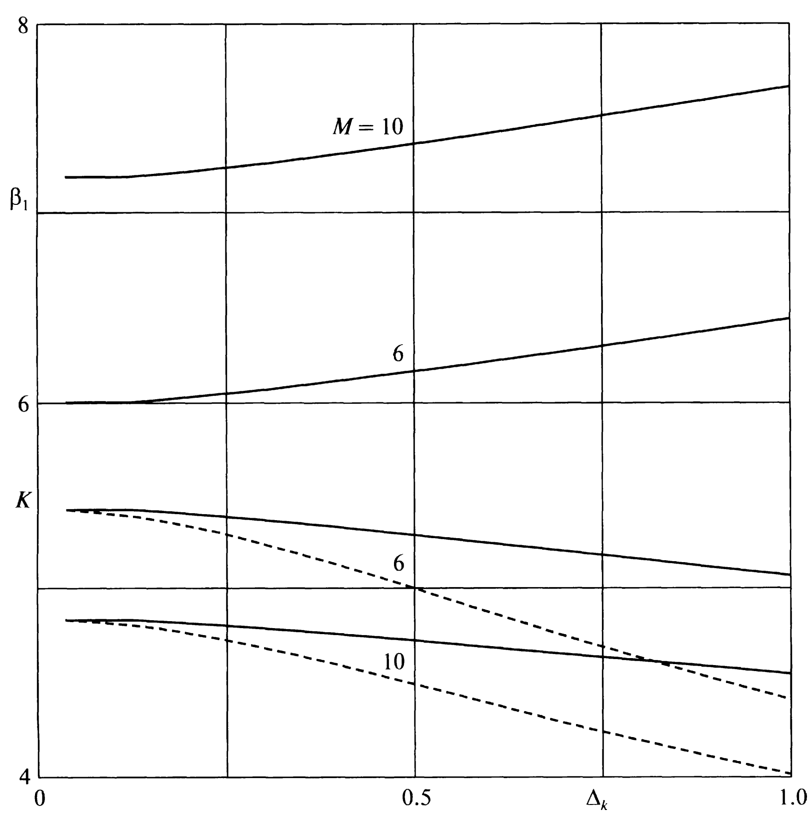
8
$M = 10$
β_1
6
6
K
6
10
4
0
0.5
Δ_k
1.0

Fig. 3.

The lift-to-drag ratio of a body designed in this manner for a fixed value of u_1 and given values of S_b and z_k is a maximum and expression (3.4) holds for it. The value of ξ_0 in formula (3.4) is found in this case taking account of relations (3.3) and (3.5) using the known parameters of the optimal shape (4.4)–(4.6), and it is determined by the quantity Δ_k.

The optimal shape of the upper part of the body is constructed under the assumption that u_1 is constant but, as was shown above, it is independent of its specific value. The lower surface of the optimal body is a segment of the plane (2.5) set at an angle of attack β_1 to the flow: $u_1 = \text{ctg}\beta_1$. The value of u_1 must ensure that the magnitude of K (3.4) is a maximum, and it is found from the equation $dK/du_1 = 0$ in which the value found from the solution of the variational problems considered above is used for ξ_0. Since the quantity ξ_0 depends on the specified isoperimetric conditions, the optimal body will have different optimal values of u_1 and K for different values of t and Δ_k.

Graphs of the optimal angle of attack β_1 and the maximum lift-to-drag ratio K against Δ_k for $C_f = 0.002$ and Mach numbers equal to 6 and 10 are shown in Fig. 3. Note that the optimal values of β_1 for which K is a maximum depend only slightly on Δ_k. For instance, when $\Delta_k \in (0,1)$, the optimal angle β_1 exceeds the magnitude of β_1^* (see the left-hand side of Table 1) by no more than 0.5°. This result is true for all values of M and C_f shown in the table which enables one to use the values of β_1^* presented in the table for an approximate estimate of the magnitude of the optimal angle.

A comparison of the shapes and values of K (3.4) of the optimal bodies designed for different isoperimetric conditions is of interest. For instance, the shape of the optimal body designed with given S_b and z_k using the values $\Delta_k = 0{,}3$, $M = 6$ and $C_f = 0.002$ is represented as a projection onto the OYZ plane by curve 1 in Fig. 4. The transverse contour of its base in a projection onto the OYZ plane is shown by the dashed line *2* in Fig. 4. For the above-mentioned parameters, the lift-to-drag ratio of the optimal body differs from the value K^* by less than 1%. The corresponding projections of the shape of the body with a linear transverse contour (4.3), designed for the same values of S_b and z_k are represented by curves *3* and *4* in Fig. 4. The length of this body is almost one and a half times greater than the length of the optimal body and its lift-to-drag ratio is less than the value of K^* by 4%. In the general case for bodies with a linear transverse contour (4.3), having the same values of S_b and z_k as the optimal bodies (4.6), graphs of the values of K against Δ_k are shown in Fig. 3 by dashed curves constructed for $C_f = 0.002$ and Mach numbers of 6 and 10.

At the same time, in the case of specified values of S_b and y_0, the optimal body is a triangular wedge with a linear transverse contour (4.3). The projection of the wing onto the OXZ plane is represented by curve *5* in Fig. 4. The wing was designed with the same values of S_b and y_0 as the body, the projection of which is represented by curve *1* in Fig. 4. The span of the wing is almost one and a half times greater than the span of the body and its lift-to-drag ratio is greater than that of a body designed with the same S_b and y_0 using formulae (4.4)–(4.6) by just 0.5%.

It should be noted that the length L of the optimal body is determined when solving the problem using the values of y_0 and u_1 obtained: $L = y_0 u_1 = y_0 \text{ctg}\beta_1$. In the case of a different area of the base S_b and the equality of z_k or y_0 of the body with the linear transverse contour

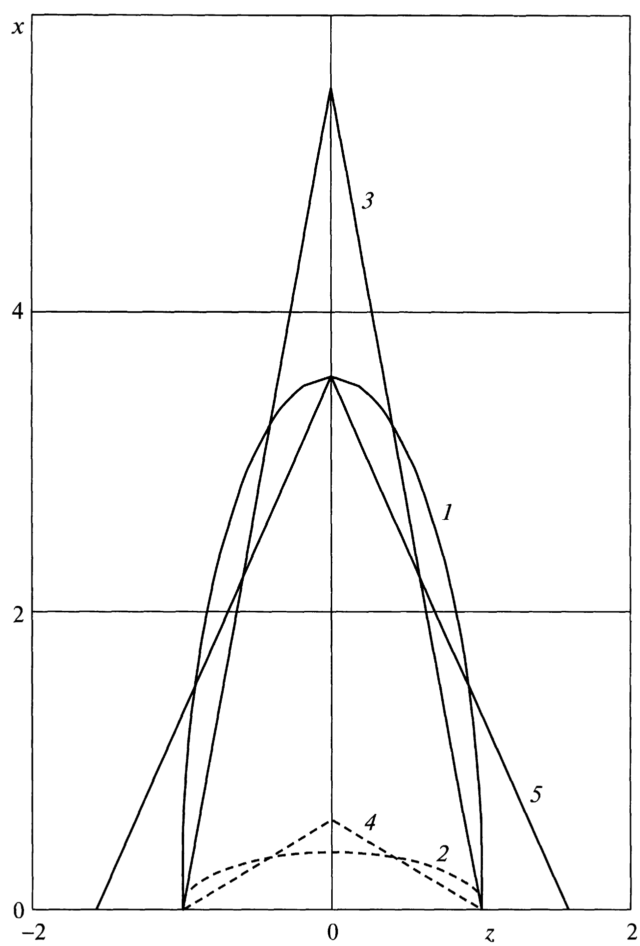

Fig. 4.

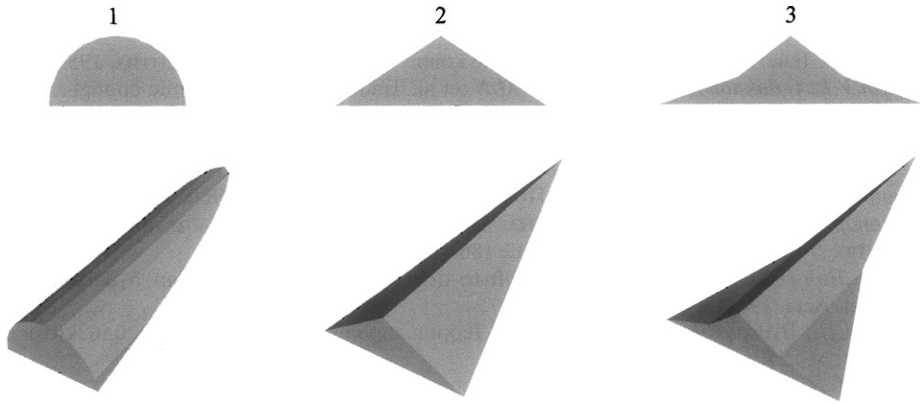

Fig. 5.

(4.3) they always have a greater length (S_b and z_k are specified) or larger span (S_b and y_0 are specified) than a body with a curvilinear transverse contour for the design of which formulae (4.4)–(4.6) were used. According to expressions (4.4), in the case of optimal bodies with a curvilinear transverse contour, the coordinate $y_k > 0$ and their shape always contains triangular lateral surfaces which are parallel to the plane of symmetry of the body.

The isoperimetric problems considered above, with a constraint on the linear dimensions of the base of the body enable one to determine its optimal shape $y = y(z)$ and the surface of the body of maximum lift-to-drag ratio. In the case when the contour of the base is given, the curve $y = y(z)$ serves as the controlling upper cylindrical surface of the body with generatrices parallel to the OX axis. In this case, the optimal value of u_1 is also found from the equation $dK/du_1 = 0$, where K is given by expression (3.4) and the quantity ξ_0, according to relation (3.3), is a known constant.

As an example, the shapes of optimal bodies with different specified shapes of the base are shown in Fig. 5: 1) semicircular, 2) triangular, and 3) triangular star, constructed for the same values of S_b and y_0: $S_b/y_0^2 = \pi/2$. The pairs of values (β_1. K) for them when $M = 15$ and $C_f = 0.002$ are as follows: 1–(8.21°, 4.398), 2–(8.09°, 4.461), 3–(8.1°, 4.454). Note that, if the shape of the bottom section is not fixed, the contour of the base of the optimal body in this case has the shape of a triangle.

5. Conclusion

As a result of the investigations carried out assuming a local character of the force interaction of the flow with an element of the body surface, shapes have been designed possessing maximum lift-to-drag ratio at supersonic flight speeds. It is shown that, when the area of the bottom section and the constant coefficient of friction are specified, the optimal body has a plane windward surface set at an angle of attack to the undisturbed flow. The leeward surface of the optimal body is parallel to the velocity vector of the undisturbed flow. The absolutely-optimal body is a two-dimensional wedge, the lift-to-drag ratio of which depends solely on the Mach number and the magnitude of the coefficient of friction. With additional constraints on the dimensions of the body, the solutions of variational problems are obtained. It is shown that, when the height of the base of the body is specified, a triangular wing with a wedge-shaped profile is optimal and, when a constraint on the span is specified, the optimal shape has plane lateral surfaces, parallel to the plane of symmetry of the body, and its leading edge and the contour of its base become curved. A comparative analysis of the optimal shapes, designed for different isoperimetric conditions, has been carried out for different Mach numbers and their lift-to-drag ratio has been determined as well as the performance of the absolutely-optimal wedge.

As regards their structure, the optimal bodies which have been found are close to bodies which are called “waveriders” in the literature (see Refs 12 and 13), the high load-bearing properties of which are well known. The upper surface of waveriders is usually specified by an undisturbed stream surface, and the lower (windward) is designed as a stream surface behind a shock wave of a given form. Plane or conical shock waves are most frequently used.[12,13] Either approximate formulae with a power dependence of the coefficient of friction on the Reynolds number[12] are used or integration is carried out over the body surface for a known state of the boundary layer,[13] in order to determine the viscous drag. In solving the problem of the shape of a waverider with the maximum lift-to-drag ratio, attention is paid to the design of the windward surface of the body and, as a rule, constraints are imposed on the class of admissible surfaces. For instance, the optimal shapes of waveriders have been found[12,13] for plane and conical shock waves in the class of power surfaces when there are certain geometrical conditions on the body. In the case of a plane shock wave, these shapes turned out to be close to a wedge with fins[12,13] and, in the case of a conical shock wave, the lower surface of the optimal waverider was found to be close to a plane.[13] It was also noted[12] that the state of the boundary layer has a minor effect on the shape of the optimal body.

In this paper, a second approach to the design of bodies possessing maximum lift-to-drag ratio was used. However, the analytical results obtained in it do not contradict the numerical results obtained for waveriders.[12,13] This provides additional arguments in favour of the chosen local approach which, as in the case of the problem of a body with minimum drag,[7–9] enables one to obtain solutions which agree with the solutions found in the case of more accurate assumptions regarding the character of the force of interaction between the flow and the body surface.

Acknowledgements

This research was supported by the Russian Foundation for Basic Research (08-01-00178, 09-01-00171a).

References

1. Chernyi GG. *Gas Flows with a High Supersonic Velocity*. Moscow: Fizmatgiz; 1959.
2. Lapygin VI, Galaktionov AYu, Kazakov N, et al. Trade-off of aerodynamic configuration for a descrete vehicle. In: *Proc. 2nd Europ. Conf. for Aerospace Sciences (EUCASS)*. 2007.
3. *Theory of Optimum Aerodynamic Shapes* Ed. by A. Miele. New York.: Acad. Press, 1965.
4. Maikapar GI. A wing with maximum lift-to-drag ratio at hypersonic velocities. *Prikl Mat Mekh* 1966;**30**(1):186–9.
5. Miele A, Hull DG, Brown SL. Maximum lift-to-drag ratio of a slender, flat-top, hypersonic body. *Astronaut Acta* 1967;**13**(2):101–7.
6. Kraiko AN, Pudovikov DYe, Yakunina GYe. *Theory of Aerodynamic Shapes Close to Optimal*. Moscow: "Yanus-K"; 2001.
7. Yakunina GYe. The design of optimal three-dimentional shapes within the limits of the local interaction model. *Prikl Mat Mekh* 2000;**64**(2):299–310.
8. Yakunina GYe. Optimal non-conical and asymmetric spatial configurations. *Prikl Mat Mekh* 2000;**64**(4):605–14.
9. Kraiko AN, Yakunina GYe. The design of optimal bodies within the limits of local interaction models. *Prikl Mat Mekh* 2008;**72**(1):41–53.
10. Huang HY. Conical bodies of given length and volume having maximum lift-to-drag ratio at hypersonic speeds. Pt 2: Variational methods. *J Astranaut Sci* 1969;**15**(3):118–23.
11. Lapygin VI. Bodies of maximum lift-to-drag ratio in hypersonic flow. *Nauchn Tr Inst Mekh MGU* 1971;**11**:44–9.
12. Keldysh VV. Optional shapes of carriers in a flow with a plane shock wave. *Izv Ross Akad Nauk Mzhg* 1996;**4**:131–41.
13. Mazhul II. A comparative analysis of the aerodynamics of waveriders constructed on the basis of conical and plane flows. *Teplofiz Aeromekh* 2007;**14**(1):99–111.

Translated by E. L. S.